Infinite Sequences and Series

Limits of Sequences

If $\lim_{n\to\infty} a_n = L$, then $\{a_n\}$ converges to L; otherwise, the sequence diverges.

Boundedness of Sequences

For all $n \geqslant N$ (where N is a nonnegative integer), $\{a_n\}$ is

- bounded above if $a_n \leqslant M$
- bounded below if $m \leqslant a_n$
- bounded if $m \leqslant a_n \leqslant M$

Increasing or Decreasing Sequences

Let N be any nonnegative integer. The sequence $\{a_n\}$ is

- increasing if $a_n < a_{n+1}$ for $n \geqslant N$
- decreasing if $a_n > a_{n+1}$ for $n \geqslant N$
- monotonic if $a_n \leqslant a_{n+1}$ or $a_n \geqslant a_{n+1}$ (nondecreasing or nonincreasing) for $n \geqslant N$

Monotonic Sequence Theorem

Any bounded, monotonic sequence is convergent.

Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Infinite Geometric Series

If $a \neq 0$ and $|r| < 1$, then $\sum_{n=0}^{\infty} ar^n = \dfrac{a}{1-r}$; if $|r| \geqslant 1$, then the series diverges.

Telescoping Series

In a telescoping series, all terms cancel out except a few—for example,

$$\begin{aligned} S_N &= \sum_{i=1}^{N} (a_i - a_{i+1}) \\ &= (a_1 - a_2) + (a_2 - a_3) + \cdots + (a_{N-1} - a_N) + (a_N - a_{N+1}) \\ &= a_1 - a_{N+1}\,. \end{aligned}$$

Then $\sum_{n=1}^{\infty} (a_n - a_{n+1}) = \lim_{N\to\infty} S_N$ if the limit exists.

p-Series

The series $\sum_{n=1}^{\infty} \dfrac{1}{n^p}$ converges for $p > 1$ and diverges for $p \leqslant 1$.

Infinite Series with Constants

Suppose that $c \neq 0$. If $\sum_{n=1}^{\infty} a_n$ converges, then so do the following series; if it diverges, then so do the following series:

$$c + \sum_{n=1}^{\infty} a_n \qquad c - \sum_{n=1}^{\infty} a_n \qquad c \sum_{n=1}^{\infty} a_n$$

Sums and Differences of Series

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge, then

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

$$\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

Integral Test

If f is continuous, positive, and decreasing on $[1, \infty)$ and $f(n) = a_n$, then

- $\sum_{n=1}^{\infty} a_n$ converges if $\int_1^{\infty} f(x)\,dx$ converges.
- $\sum_{n=1}^{\infty} a_n$ diverges if $\int_1^{\infty} f(x)\,dx$ diverges.

Direct Comparison Test

Let a_n and b_n both be positive.

- If $a_n \leqslant b_n$ and $\sum b_n$ converges, then $\sum a_n$ also converges.
- If $a_n \geqslant b_n$ and $\sum b_n$ diverges, then $\sum a_n$ also diverges.

Limit Comparison Test

Let a_n and b_n both be positive, with $L = \lim_{n\to\infty} \dfrac{a_n}{b_n}$. If L is positive and finite, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Alternating Series Test

The series $\sum_{n=1}^{\infty} (-1)^n b_n$, $b_n > 0$, converges if $\lim_{n\to\infty} b_n = 0$ and $b_{n+1} \leqslant b_n$ for all $n \geqslant 1$.

Infinite Sequences and Series

Absolute and Conditional Convergence

- If $\sum |a_n|$ converges, then $\sum a_n$ is absolutely convergent (and therefore converges).
- If $\sum a_n$ converges but $\sum |a_n|$ diverges, then $\sum a_n$ is conditionally convergent.

Ratio Test

Let $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$.

- If $L < 1$, then $\sum a_n$ converges absolutely.
- If $L > 1$, then $\sum a_n$ diverges.
- If $L = 1$, then the test is inconclusive.

Root Test

Let $L = \lim_{n\to\infty} |a_n|^{1/n}$.

- If $L < 1$, then $\sum a_n$ converges absolutely.
- If $L > 1$, then $\sum a_n$ diverges.
- If $L = 1$, then the test is inconclusive.

Differentiating and Integrating Power Series

If f has a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ with radius of convergence R, then

$$f'(x) = \sum_{n=1}^{\infty} n\, c_n (x-a)^{n-1}$$

$$\int f(x)\,\mathrm{d}x = C + \sum_{n=0}^{\infty} \frac{c_n(x-a)^{n+1}}{n+1}$$

Both series have a radius of convergence of R.

Taylor and Maclaurin Series

If f has derivatives of all orders at a, then the Taylor series of f centered at a is

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

If $a = 0$, then the Maclaurin series of f is

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

(Assume $0! = 1$.)

Binomial Series

If k is any real number and $|x| < 1$, then

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$
$$= \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

where $$\binom{k}{n} = \frac{k(k-1)(k-2)\cdot\cdots\cdot(k-n+1)}{n!}$$

Error Bounds with the Integral Test

Suppose that $S = \sum_{n=1}^{\infty} a_n$ is approximated by $S_N = \sum_{i=1}^{N} a_i$. If f is a continuous, positive, and decreasing function on $[N, \infty)$ with $f(n) = a_n$, then the error in the estimate, $R_N = S - S_N$, satisfies

$$\int_{N+1}^{\infty} f(x)\,\mathrm{d}x \leqslant R_N \leqslant \int_{N}^{\infty} f(x)\,\mathrm{d}x$$

Alternating Series Error Bound

If a convergent series $S = \sum_{n=1}^{\infty} (-1)^n\, b_n$ $(b_n > 0)$ is approximated by $S_N = \sum_{i=1}^{N} (-1)^i\, b_i$, and $\lim_{n\to\infty} b_n = 0$ and $b_{n+1} \leqslant b_n$, then the error in the estimate satisfies

$$|S - S_N| \leqslant b_{N+1}$$

Taylor's Remainder Theorem

If f has derivatives up to order $N+1$ and $\left|f^{(N+1)}(x)\right| \leqslant M$ for $|x-a| \leqslant r$, then the remainder $R_N(x)$ of the Nth-degree Taylor polynomial centered at a satisfies

$$|R_N(x)| \leqslant \frac{M}{(N+1)!}|x-a|^{N+1}$$

Maclaurin Series of Key Functions

Series	Convergence
$\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	$-1 < x < 1$
$e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$	$-\infty < x < \infty$
$\sin x = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$	$-\infty < x < \infty$
$\cos x = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{(2n)!} = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$	$-\infty < x < \infty$
$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{x^n}{n} = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$	$-1 < x \leqslant 1$
$\tan^{-1} x = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{2n+1} = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots$	$-1 \leqslant x \leqslant 1$
$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \dfrac{k(k-1)}{2!} x^2 + \dfrac{k(k-1)(k-2)}{3!} x^3 + \cdots$	$-1 < x < 1$*

**The convergence at the endpoints $x = \pm 1$ depends on the value of k.*